Orbits on Lagrangian Grassmanian

Hongyu He
Department of Mathematics,
Louisiana State University,
Baton Rouge, LA 70803, U.S.A.
email: livingstone@alum.mit.edu

The purpose of this note is to give a classification of the orbital structure of certain reductive group actions on the Lagrangian Grassmanian. The groups under consideration are $Sp \times Sp$ and GL. In general, the finiteness of orbits is known ([8], [1], [9], [2], [10]). However, the homogeneous structure of each individual orbit is not known except perhaps the generic open orbit [1]. The main results are given in Theorem 2.2 and Theorem 3.2. Some of the results here will be used to analyze the representations of the universal covering of the symplectic group on the Shilov boundary.

1 Definitions

Let (V, Ω) be a symplectic space of dimension $2m$. Let U be a subspace of V. Define

$$U^{\perp} = \{v \in V \mid \Omega(v, U) = 0\}.$$

Sometimes, we also write it as $U^{\perp,\Omega}$. A subspace U of V is called isotropic if and only if $U^{\perp} \supseteq U$. Equivalently, U is isotropic if $\Omega|_U$ is zero. An isotropic subspace of V is called a Lagrangian subspace if $\dim U = \frac{\dim V}{2}$. Obviously, $\frac{\dim V}{2}$ is the maximal dimension of an isotropic subspace. So U is Lagrangian if and only if $U^{\perp} = U$.

Let $Sp(V)$ be the symplectic group. Let $Iso(V, i)$ be the set of i-dimensional isotropic subspaces of V. Then $Sp(V)$ acts on $Iso(V, i)$ transitively. Fix a base point $U \in Iso(V, i)$. Let P_U be the stablizer of U. Then P_U is a maximal parabolic subgroup and $Iso(V, i) \cong Sp(V)/P_U$.

Let $\mathcal{L}(V)$ be the set of Lagrangian subspaces of V. $\mathcal{L}(V)$ is called the Lagrangian Grassmanian. Let U be a base point in $\mathcal{L}(V)$. Then the stabilizer P_U is isomorphic to the Siegle parabolic subgroup and $\mathcal{L}(V) \cong Sp(V)/P_U$.

Unless stated otherwise, all vector spaces and groups are over $\mathbb{R}$.

[1]Wee-Teck Gan just informed me that the stabilizer for $Sp \times Sp$ action was computed by Ikeda, see Lemma 3.1 in [7] and the paragraph after it

2 $Sp \times Sp$ action on Lagrangian Grassmanian

Let m, n be two positive integers and $m \leq n$. Let (V_1, Ω_1) be a symplectic space of $2m$ dimension and (V_2, Ω_2) be a symplectic space of $2n$ dimension. Let $V = V_1 \oplus V_2$ and $\Omega = \Omega_1 - \Omega_2$. Then (V, Ω) is a symplectic space. Let P_1 be the canonical projection from V to V_1 and P_2 be the canonical projection from V to V_2. In this section, we are interested in the orbital structure of $Sp(V_1) \times Sp(V_2)$ on $\mathcal{L}(V_1 \oplus V_2)$. The classification of $Sp(V_1) \times Sp(V_2)$-orbits is well-known (see [3], [4], [8]). The isotropic group is obtained in Theorem 2.2.

Let $U \in \mathcal{L}(V)$. Let $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$. Then
- U_1 is an isotropic subspace in (V_1, Ω_1).
- U_2 is an isotropic subspace in (V_2, Ω_2).

Lemma 2.1 *Let $U \in \mathcal{L}(V)$. Then $P_1(U)^{\perp, \Omega_1} = U_1$ and $P_2(U)^{\perp, \Omega_2} = U_2$.*

Proof: Suppose that $x \in V_1$ and $\Omega_1(x, P_1(U)) = 0$. Then $\Omega(x, P_1(U)) = 0$. Clearly $\Omega(x, V_2) = 0$. So $\Omega(x, V_2 + P_1(U)) = 0$. Notice that $u - P_1(u) \in V_2$ for any $u \in U$. Hence $\Omega(x, U) = 0$. This implies that $x \in U$. It follows that $x \in U \cap V_1 = U_1$. The converse is also true, that is, $\Omega_1(U_1, P_1(U)) = 0$. This follows directly from $\Omega(U_1, U) = 0$. We have thus shown that $P_1(U)^{\perp, \Omega_1} = U_1$. The second statement follows similarly. $\square$

Corollary 2.1 *The null space of $\Omega_1|_{P_1(U)} = U_1$. The null space of $\Omega_2|_{P_2(U)} = U_2$.*

Corollary 2.2 *$\Omega_1|_{P_1(U)}$ reduces to a nondegenerate form Ω_1' on $P_1(U)/U_1$. $\Omega_2|_{P_2(U)}$ reduces to a nondegenerate form Ω_2' on $P_2(U)/U_2$. Furthermore,*

$$P_1(U)/U_1 \cong U/(U_1 \oplus U_2) \cong P_2(U)/U_2.$$

Proof: The first two statements are obvious. Since $ker P_1 = V_2$. Therefore, $P_1(U) \cong U/U \cap V_2 = U/U_2$. Our assertion follows immediately. $\square$

Corollary 2.3 $\dim(U_2) = \dim(U_1) + n - m$.

Proof: Since $U_1 = P_1(U_1)^{\perp, \Omega_1}$, $\dim U_1 + \dim(P_1(U_1)) = 2m$. Notice that

$$\dim(P_1(U_1)) = \dim(U/U_2) = \dim U - \dim(V_2 \cap U) = (n + m) - (\dim U_2).$$

Hence, $\dim U_1 + n + m - \dim U_2 = 2m$. Consequently, $\dim U_1 = \dim U_2 + m - n$. $\square$

Theorem 2.1 ([3], [8]) *Let $\mathcal{L}_i$ be the set of Lagrangian subspaces of (V, Ω) such that*

$$\dim(U \cap V_1) = i.$$

Then $\dim(U \cap V_2) = i + (n - m)$ and $\mathcal{L}_i$ is a single $Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})$ orbit.

Various form of this theorem can be found in a number of papers. Some come with a proof and some do not come with a proof.

Theorem 2.2 *Let $i \leq m$. Let $P_{i,2m-2i}$ be a maximal parabolic subgroup of $Sp_{2m}(\mathbb{R})$ preserving an i-dimensional isotropic subspace. Let $P_{n-m+i,2m-2i}$ be a maximal parabolic subgroup of $Sp_{2n}(\mathbb{R})$ preserving an $n-m+i$-dimensional isotropic subspace. Let $GL_i Sp_{2m-2i} N_{i,2m-2i}$ be the Langlands decomposition of $P_{i,2m-2i}$. Let $GL_{n-m+i} Sp_{2m-2i}(\mathbb{R}) N_{n-m+i,2m-2i}$ be the Langlands decomposition of $P_{n-m+i,2m-2i}$. Let*

$$H_i = \{(m_1 u n_1, m_2 {}^t u^{-1} n_2) \mid m_1 \in GL_i, m_2 \in GL_{n-m+i}, u \in Sp_{2m-2i}(\mathbb{R}), n_1 \in N_{i,2m-2i}, n_2 \in N_{n-m+i,2m-2i}\}.$$

Then $\mathcal{L}_i \cong Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})/H_i$. In particular, if $m \leq n$, then

$$\mathcal{L}_0 \cong Sp_{2n}(\mathbb{R})/GL_{n-m} N_{n-m,2m}.$$

Proof of Theorem 2.1 and Theorem 2.2: It suffices to prove that $\mathcal{L}_i$ is a single $Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})$-orbit. For every $U \in \mathcal{L}_i$, let $\pi(U) = (U_1, U_2)$. Then $\dim U_1 = i$ and $\dim U_2 = n-m+i$. Clearly, π is a map from $\mathcal{L}_i$ to $Iso(2m, i) \times Iso(2n, i+n-m)$. We claim that π is a projection. Let us fix U_1, U_2. Consider

$$\mathcal{L}_{U_1,U_2} = \{U \in \mathcal{L}_i \mid U \cap V_1 = U_1, U \cap V_2 = U_2\}.$$

We will show that $\mathcal{L}_{U_1,U_2}$ is nonempty and isomorphic to $Sp_{2m-2i}(\mathbb{R})$.

Since U_1, U_2 are fixed, $P_1(U) = U_1^{\perp,\Omega_1}$ and $P_2(U) = U_2^{\perp,\Omega_1}$ are known. In addition

$$U \subseteq P_1(U) \oplus P_2(U) = U_1^{\perp,\Omega_1} \oplus U_2^{\perp,\Omega_1}.$$

Clearly, $U \in \mathcal{L}_{U_1,U_2}$ is uniquely determined by $U/U_1 \oplus U_2$, a $2m-2i$ dimensional subspace in

$$U_1^{\perp,\Omega_1} \oplus U_2^{\perp,\Omega_1}/(U_1 \oplus U_2),$$

and vice versa. Let $u, w \in U$. We compute

$$0 = \Omega(u,w) = \Omega_1(P_1(u), P_1(w)) - \Omega_2(P_2(u), P_2(w)) = \Omega_1'([P_1(u)], [P_1(w)]) - \Omega_2'([P_2(u)], [P_2(w)]).$$

Here $[P_i(u)], [P_i(w)] \in U_i^{\perp,\Omega_i}/U_i$ and Ω_i' is the reduced symplectic form on $U_i^{\perp,\Omega_i}/U_i$. For each $U \in \mathcal{L}_{U_1,U_2}$, define $P_i' : U/U_1 \oplus U_2 \to U_i^{\perp,\Omega_i}/U_i$ by letting $P_i'([u]) = [P_i(u)]$. We see that $U \in \mathcal{L}_{U_1,U_2}$ if and only if for every $[u], [w] \in U/U_1 \oplus U_2$,

$$\Omega_1'(P_1'([u]), P_1'([w])) = \Omega_2'(P_2'([u]), P_2'[w]).$$

Notice that P_i' is injective. An easy computation shows that

$$\dim(U/U_1 \oplus U_2) = 2m - 2i = \dim(U_1^{\perp,\Omega_1}/U_1) = \dim(U_2^{\perp,\Omega_2}).$$

So P_i' is surjective. It follows that $U \in \mathcal{L}_{U_1,U_2}$ is in one-to-one correspondence with $P_2'(P_1')^{-1}$, a symplectic isomorphism between $(U_1^{\perp,\Omega_1}/U_1, \Omega_1')$ and $(U_2^{\perp,\Omega_2}/U_2, \Omega_2')$. The correspondence is explicitly given by

$$\begin{aligned} \phi \in Sp(U_1^{\perp,\Omega_1}/U_1, U_2^{\perp,\Omega_2}/U_2) &\to \{([u_1], \phi([u_1]) \in U_1^{\perp,\omega_1}/U_1 \oplus U_2^{\perp,\Omega_2}/U_2\} \\ &\to \{(u_1,u_2) \mid u_1 \in U_1^{\perp,\Omega_1}, u_2 \in U_2^{\perp,\Omega_2}, u_2 + U_2 = \phi(u_1+U_1)\}. \end{aligned} \tag{1}$$

We have thus obtained a principal fibration

$$Sp_{2m-2i}(\mathbb{R}) \to \mathcal{L}_i \to Iso(2m,i) \times Iso(2n, n-m+i). \tag{2}$$

Observe that the stabilizer of U_1 in $Sp(V_1)$, P_{U_1}, acts on U_1 and $U_1^{\perp,\Omega_1}$. P_{U_1} is of the form $Sp_{2m-2i}(\mathbb{R})GL_iN_{i,2m-2i}$. It acts on $U_1^{\perp,\Omega_1}/U_1$ by the $Sp_{2m-2i}(\mathbb{R})$ factor. By the correspondence 1, P_{U_1} acts on $\mathcal{L}_{U_1,U_2}$ transitively. It follows that $\mathcal{L}_i$ is a single $Sp_{2m}(\mathbb{R}) \times Sp_{2n}(\mathbb{R})$ orbit. This is Theorem 2.1.

Notice that $Iso(2m,i) \cong Sp_{2m}(\mathbb{R})/P_{i,2m-2i}$ and $Iso(2n, n-m+i) \cong Sp_{2n}(\mathbb{R})/P_{n-m+i,2m-2i}$. Theorem 2.2 follows from the principal fibration (2). $\square$

The following corollaries are more or less obvious.

Corollary 2.4 *Let $0 \leq i \leq m$. $U \in \mathcal{L}_i$ if and only if there exist $U_1 \in Iso(2m,i)$, $U_2 \in Iso(2n, n-m+i)$and a symplectic isomorphism between $(U_1^{\perp,\Omega_1}/U_1, \Omega_1')$ and $({U_2}^{\perp,\Omega_2}/U_2, \Omega_2')$ such that $U_1 = U \cap V_1$ and $U_2 = U \cap V_2$ and*

$$U = \{(u_1,u_2) \in U_1^{\perp,\Omega_1} \oplus U_2^{\perp,\Omega_2} \mid \phi(u_1+U_1) = u_2 + U_2\}.$$

Here Ω_i' is the reduced symplectic form on ${U_i}^{\perp,\Omega_i}/U_i$.

Corollary 2.5 *Let $i \leq m \leq n$. Then $\dim(\mathcal{L}_i) = \frac{(n+m+1)(n+m)}{2} - i^2 - ni + mi$. There are total of $m+1$ $Sp_{2n}(\mathbb{R}) \times Sp_{2m}(\mathbb{R})$ orbits in $\mathcal{L}(V)$ and*

$$\mathcal{L}_{i+1} \subseteq cl(\mathcal{L}_i).$$

3 GL_n-orbits

Let Ω be a symplectic form on a $2n$ dimensional real vector space V. Let $\mathcal{L}$ be the Lagrangian Grassmannian. Fix two Lagrangian subspaces W_1 and W_2 such that

$$W_1 \oplus W_2 = V.$$

Clearly, Ω identifies W_2 with W_1^*, the dual of W_1. Let GL_n be the subgroup of $Sp(V,\Omega)$ preserving W_1 and W_2. Let $g \in GL_n$. Then g acts on $U \in \mathcal{L}$ by

$$gU = \{(gx_1, g^*x_2) \mid (x_1,x_2) \in U\}.$$

Now we want to study the GL_n orbits on $\mathcal{L}$.

Let $e_1, e_2, \ldots e_n$ be a basic for W_1. Let $f_1, f_2, \ldots, f_n$ be a basis for W_2 such that $\Omega(e_i, f_j) = \delta_i^j$ where δ_i^j is the Kronecker symbol. Clearly, f_i can be identified with e_i^*. Define $J \in End(V)$ by setting

$$Je_i = -f_i, \qquad Jf_i = e_i.$$

Then

$$\Omega(x, y) = (x, Jy) = -(Jx, y),$$

where $(,)$ is the standard inner product. Furthermore $Jgx_1 = g^* Jx_1$.

Put $U_1 = U \cap W_1$ and $U_2 = U \cap W_2$. Let

$$\mathcal{L}_{i,j} = \{U \in \mathcal{L} \mid \dim(U \cap W_1) = i, \dim(U \cap W_2) = j\}.$$

We have

Lemma 3.1 *$\mathcal{L}_{0,0}$ consists of $n+1$ GL_n orbits:*

$$\mathcal{L}_{0,0}^0, \ldots, \mathcal{L}_{0,0}^i, \ldots, \mathcal{L}_{0,0}^n,$$

where $\mathcal{L}_{0,0}^i \cong GL_n/O(i, n-i)$.

Proof: For every $U \in \mathcal{L}_{0,0}$, define $\phi_U(x_1) = x_2$ if and only if $(x_1, x_2) \in U$. Since $\dim U = \dim W_1 = \dim W_2 = n$ and $\dim U \cap W_1 = \dim U \cap W_2 = 0$, ϕ_U is a well-defined isomorphism from W_1 to W_2. Intuitively, U is the graph of ϕ_U. Let $(x_1, \phi_U(x_1)), (y_1, \phi_U(y_2)) \in U$. Then

$$\begin{aligned} 0 =&\Omega((x_1, \phi_U(x_1)), (y_1, \phi_U(y_1)) = \Omega(x_1, \phi_U(y_1)) + \Omega(\phi_U(x_1), y_1) \\ =&\Omega(x_1, \phi_U(y_1)) - \Omega(y_1, \phi_U(x_1)) = (x_1, J\phi_U(y_1)) - (y_1, J\phi_U(x_1)). \end{aligned} \tag{3}$$

Therefore $(x_1, J\phi_U(y_1)) = (y_1, J\phi_U(x_1))$ for all $x_1, y_1 \in W_1$. It follows that $(x_1, J\phi_U(y_1))$ is a Hermitian form on W_1.

Let $g \in GL_n$. Then

$$gU = \{(gx_1, g^*\phi_U(x_1) \mid x_1 \in W_1\} = \{(x_1, g^*\phi_U g^{-1}x_1) \mid x_1 \in W_1\}.$$

So $\phi_{gU} = g^*\phi_U g^{-1}$. The Hermitian form

$$(x_1, J\phi_{gU}(y_1)) = (x_1, Jg^*\phi_U g^{-1}y_1) = (x_1, g^* J\phi_U g^{-1}y_1).$$

Clearly, GL_n acts on the non-degenerate Hermitian forms with $n+1$ orbits. Each orbit consists of Hermitian forms with a fixed signature. Let $\mathcal{L}_{0,0}^i$ be the set of U such that $(x_1, J\phi_U(y_1))$ has signature $(i, n-i)$. Then $\mathcal{L}_{0,0}^i$ is a single GL_n orbit and $\mathcal{L}_{0,0}^i \cong GL_n/O(i, n-i)$. $\square$.

The case $(\dim U_1, \dim U_2) \neq (0, 0)$ is considerably more difficult. One needs to carry out two symplectic reductions. However, the ending result is easy to state.

Theorem 3.1 *Each $\mathcal{L}_{i,j}$ consists of $n-i-j+1$ GL_n-orbits.*

We shall now give a proof of this theorem and the structure of each orbit will be stated at the end of this discussion.

Let $U \in \mathcal{L}_{i,0}$. Let $U_1 = U \cap W_1$. Then U_1 is an isotropic subspace. Fix an $U_1 \subseteq W_1$ and let

$$\mathcal{L}_{U_1,0} = \{U \in \mathcal{L} \mid U \cap W_1 = U_1, U \cap W_2 = 0\}.$$

We have the fiberation

$$\mathcal{L}_{U_1,0} \to \mathcal{L}_{i,0} \to G(n,i),$$

where $G(n,i)$ is the i-th Grassmanian of W_1. Notice that GL_n acts on $G(n,i)$. Let P_{U_1} be the stabilizer of $U_1 \in G(n,i)$. The P_{U_1} is isomorphic to the block-wise upper triangular matrices of size $(n-i,i)$. It suffices to study the action of P_{U_1} on $\mathcal{L}_{U_1,0}$.

Consider now $U_1^{\perp}$, with respect to Ω. Let Ω_1' be the reduced symplectic form on $U_1^{\perp}/U_1$. Since $U_1 \subseteq U$, $U \subseteq U_1^{\perp}$. We see that

$$[U]^{\perp,\Omega_1'} = U^{\perp}/U_1 = U/U_1.$$

So U/U_1 is a Lagrangian subspace of $U_1^{\perp}/U_1$. Then $\mathcal{L}_{U_1,0}$ can be identified with a subset of $\mathcal{L}(U_1^{\perp}/U_1)$. Furthermore, P_{U_1} acts on U_1 and $U_1^{\perp}$. The $GL(i)$ factor acts on U_1 naturally. The nilradical N acts on $U_1^{\perp}$ by shifts along directions in U_1. We thus obtain a GL_{n-i} action on $\mathcal{L}(U_1^{\perp}/U_1)$. By choosing U_1 to be the linear space spanned by $e_1, \ldots, e_i$, the GL_{n-i} action on $\mathcal{L}(U_1^{\perp}/U_1)$ can be identified with the standard GL_{n-i} action on $\mathcal{L}(\mathbb{R}^{2n-2i})$.

Let $W_2' = W_2 \cap U_1^{\perp}$. Then $(W_2')^{\perp} = W_2 \oplus U_1$ and $\dim(W_2') = n-i$. Furthermore

$$(W_2' \oplus U_1)^{\perp} = (W_2 \oplus U_1) \cap U_1^{\perp} = W_2' \oplus U_1.$$

We see that $W_2' \oplus U_1$ is a Lagrangian subspace in $U_1^{\perp}/U_1$. Since $W_2 \cap U = 0$, $W_2' + U_1 \cap U = U_1$. Since $W_1 \subseteq U_1^{\perp}$ and W_1 is Lagrangian, W_1/U_1 is a Lagrangian subspace of $U_1^{\perp}/U_1$. Furthermore,

$$U_1^{\perp}/U_1 = (W_2' + U_1)/U_1 \oplus W_1/U_1.$$

Finally, notice that $U/U_1 \cap (W_2' + U_1)/U_1 = [0]$ and $U/U_1 \cap W_1/U_1 = [0]$. We see that

$$\mathcal{L}_{U_1,0} \cong \mathcal{L}_{0,0}(\Omega_1', (W_2' + U_1)/U_1 \oplus W_1/U_1).$$

By Lemma 3.1, we obtain

Lemma 3.2 *$\mathcal{L}_{i,0}$ consists of $n-i+1$ GL_n orbits:*

$$\mathcal{L}_{i,0}^{0}, \ldots, \mathcal{L}_{i,0}^{n-i}.$$

Let P_i be the maximal parabolic subgroup with Langlands decomposition $GL_i GL_{n-i} N_{i,n-i}$. Then $\mathcal{L}_{i,0}^{j} \cong GL_n / GL_i O(j, n-i-j) N_{i,n-i}$.

If $ij \neq 0$, we are yet to apply another symplectic reduction. Fix a U_2, a j-dimensional subspace of W_2. U_2 is isotropic automatically. Let $\mathcal{L}_{i,U_2}$ be the set of Lagrangian subspaces satisfying

$$U \cap W_2 = U_2, \qquad \dim(U \cap W_1) = i.$$

Then we have a fibration

$$\mathcal{L}_{i,U_2} \to \mathcal{L}_{i,j} \to G(n,j).$$

Let P_{U_2} be the stabilizer of U_2 in GL_n. Then P_{U_2} is isomorphic to $GL_j GL_{n-j} N_{j,n-j}$.

Again, we construct the reduced symplectic form Ω_2 on $U_2^{\perp}/U_2$. Let $W_1' = W_1 \cap U_2^{\perp}$. Then $(W_1')^{\perp} = W_1 \oplus U_2$ and $\dim(W_1') = n - j$. Notice that

$$(W_1' \oplus U_2)^{\perp} = (W_1 \oplus U_2) \cap U_2^{\perp} = W_1' \oplus U_2.$$

We see that $(W_1' + U_2)/U_2$ is a Lagrangian subspace of $U_2^{\perp}/U_2$. W_2/U_2 is also a Lagrangian subspace of $U_2^{\perp}/U_2$. In addition

$$U_2^{\perp}/U_2 = (W_1' + U_2/U_2) \oplus W_2/U_2.$$

The group P_{U_2} acts on $U_2^{\perp}/U_2$ via the factor GL_{n-j}.

Since U is Lagrangian, U/U_2 is a Lagrangian subspace in $U_2^{\perp}/U_2$. Furthermore $\dim(U/U_2 \cap W_2/U_2) = 0$. Observe that

$$U/U_2 \cap (W_1' + U_2/U_2) = U \cap (W_1' + U_2)/U_2 = (U \cap W_1') + U_2/U_2$$

and

$$(U \cap W_1')^{\perp} = U^{\perp} + (W_1')^{\perp} = U + W_1 + U_2 = U + W_1 = (U \cap W_1)^{\perp}.$$

So $U \cap W_1' = U \cap W_1 = U_1$. It follows that

$$U/U_2 \cap (W_1' + U_2/U_2) = (U \cap W_1') + U_2/U_2 = U_1 + U_2/U_2.$$

Consequently, $\dim(U/U_2 \cap W_1' + U_2/U_2) = i$. Now we can identify $\mathcal{L}_{i,U_2}$ with

$$\mathcal{L}_{i,0}(U_2^{\perp}/U_2, W_2/U_2 \oplus W_1' + U_2/U_2)$$

by

$$U \leftrightarrow U/U_2.$$

By Lemma 3.2, GL_{n-j} acts on $\mathcal{L}_{i,0}(U_2^{\perp}/U_2)$ with $n-j-i+1$-orbits:

$$\mathcal{L}_{i,0}^{0}(U_2^{\perp}/U_2), \ldots, \mathcal{L}_{i,0}^{i}(U_2^{\perp}/U_2)$$

and

$$\mathcal{L}_{i,0}^{k}(U_2^{\perp}/U_2) \cong GL_{n-j}/GL_i O(k, n-i-j-k) N_{n-i-j,i}.$$

To summarize, we have the following theorem

Theorem 3.2 *Each $\mathcal{L}_{i,j}$ consists of $n-i-j+1$ GL_n-orbits*

$$\mathcal{L}_{i,j}^{0}, \ldots \mathcal{L}_{i,j}^{n-i-j}.$$

Let $P_{i,j,n-i-j}$ be the block-wise upper triangular matrices with size $(i,j,n-i-j)$. Let N be the nilradical. Then

$$\mathcal{L}_{i,j}^{k} \cong GL_n/GL_jGL_iO(k,n-i-j-k)N.$$

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